Rehabilitation and Law Enforcement as Optimal Controls in a Mathematical Model of Social Behavior

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ABSTRAK

Perilaku sosial merupakan hasil interaksi antarindividu yang dapat membentuk kecenderungan menuju perilaku positif maupun menyimpang. Berdasarkan sudut pandang ini, dikembangkan suatu model matematika perilaku sosial yang membagi populasi menjadi kelompok kriminal dan non-kriminal. Penelitian-penelitian sebelumnya umumnya hanya mempertimbangkan penegakan hukum, seperti penangkapan dan pemenjaraan, sebagai strategi penanganan terhadap perilaku menyimpang, tanpa memasukkan aspek rehabilitasi. Penelitian ini mengkaji penerapan kendali optimal dalam model matematika perilaku sosial guna meminimalkan jumlah individu dalam kelompok kriminal, dengan rehabilitasi dan penegakan hukum sebagai variabel kendali. Permasalahan kendali optimal diselesaikan menggunakan Prinsip Minimum Pontryagin, sedangkan simulasi numerik dilakukan dengan metode Forward-Backward Sweep. Hasil simulasi menunjukkan bahwa kombinasi strategi rehabilitasi dan penegakan hukum secara signifikan mampu menurunkan populasi kriminal dalam model.

Kata kunci: penegakan hukum; perilaku sosial; rehabilitasi

ABSTRACT

Social behavior is the result of interactions between individuals, which can lead to tendencies toward either positive or deviant behavior. From this perspective, a mathematical model of social behavior is developed, dividing the population into criminal and non-criminal groups. Previous studies generally considered only law enforcement strategies—such as arrest and imprisonment—as responses to deviant behavior, without incorporating the aspect of rehabilitation. This study examines the application of optimal control in a mathematical model of social behavior to minimize the number of individuals in the criminal group, using rehabilitation and law enforcement as control variables. The optimal control problem is solved using Pontryagin's Minimum Principle, and numerical simulations are performed using the Forward-Backward Sweep Method. The simulation results show that a combination of rehabilitation and law enforcement strategies can significantly reduce the criminal population in the model.

Keywords: *Law enforcement; Rehabilitation; Social behavior*

INTRODUCTION

Social behavior encompasses all forms of actions or responses performed by individuals in the context of interactions with other individuals or groups within society. This behavior reflects how a person acts, communicates, or responds to the social norms and values around them. Social behavior and criminal acts are closely related, as criminal acts often arise as deviations from the prevailing social norms within a society. A criminal act is a form of deviant behavior that violates formal laws (legislation). It is a type of social behavior, but one that is negative, as it harms others and disrupts social order. A simple scheme illustrating the relationship between social behavior and criminal acts is shown in Figure 1.

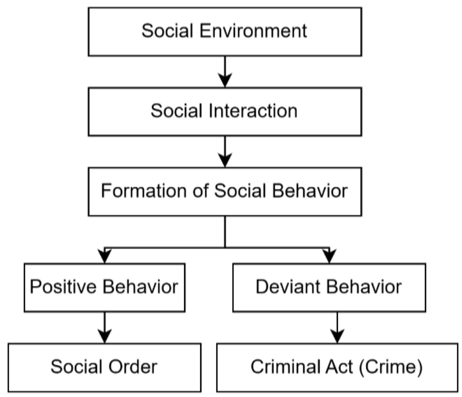

Figure 1. Diagram of the Relationship Between Social Behavior and Criminal Acts

One of the criminological theories from a social perspective, the Differential Association Theory, states that no behavior is inherited from one's parents. Patterns of criminal behavior are not passed down genetically but are learned through social interaction. Criminal behavior is acquired within groups through interaction and communication, and it is learned in those group settings [1]. Several studies in Indonesia that have discussed this theory include research on acts of terrorism [1], violence [2], cyberbullying [3], juvenile delinquency behavior such as "klitih" [4], cigarette use among teenagers [5], hacking cases [6], and drug abuse [7].

Social behavior related to criminal acts can be studied not only from a social science perspective. Using a mathematical approach, researchers have also examined social behavior, with mathematics serving as an alternative tool in combating crime. These studies include topics such as gang membership [8], financial crimes [9], armed groups [10], corruption [11], bullying [12], and drug abuse [13]. Some studies do not focus on a specific type of criminal act. For example, in the study by Abbas et al. [14], a population is divided into two compartments: criminal minded and non-criminal minded populations. In that study, law enforcement is applied as a means to reduce criminal behavior. This work was further developed in subsequent studies. Srivastav et al. [15] and Tripathi et al. [16] expanded the model by incorporating a Holling type II response function to describe interactions between criminal minded and non-criminal minded populations. Kumar & Abbas [17] and Kumar et al. [18] enhanced the model by considering age structure into the compartmental diagram. Izzati et al. [19] further extended the model by adding a religious population compartment.

In the study by Abbas et al. [14], only law enforcement was implemented as a strategy to reduce the criminal minded population—rehabilitation programs were not included. Law enforcement generally refers to the efforts of the state or legal authorities (such as the police, prosecutors, and courts) to ensure the enforcement of laws. This includes the arrest of offenders, judicial processes, sentencing, and execution of punishment (such as imprisonment or fines). On the other hand, rehabilitation is a more restorative approach that focuses on reintegrating offenders into society as law-abiding individuals. Rehabilitation is commonly applied to prisoners (e.g., vocational training, counseling), drug addicts (through medical and social rehabilitation), and juvenile offenders (via restorative justice approaches). In certain contexts, rehabilitation can be part of law enforcement, particularly in legal systems emphasizing corrective or restorative justice rather than purely punitive measures. For example, drug rehabilitation as an alternative to imprisonment, or rehabilitation within correctional institutions as part of prisoner reformation. However, not all forms of rehabilitation are considered law enforcement—for instance, purely medical rehabilitation (such as for accident victims) is not part of the legal system. In this study, rehabilitation programs are considered as an intervention for individuals who have committed criminal acts. From this foundation, the study explores an optimal control problem applied to a mathematical model of social behavior, where law enforcement and rehabilitation serve as the control variables.

METHOD

The steps carried out in this study are illustrated in the flowchart in Figure 2.

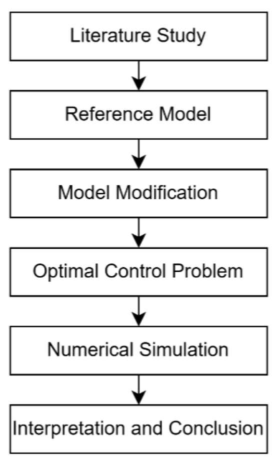

Figure 2. Research Flowchart

The first step is a literature review. In this phase, materials related to social behavior are collected, including both social science theories and mathematical models that researchers have previously discussed. The second step is to determine the model to be modified. This study builds upon the model developed by Abbas et al. [14], which models the social interaction between the criminal and non-criminal populations as follows:

$$\frac{dN_p}{dt} = \mu N_p \left(1 - \frac{N_p}{K}\right) - \alpha N_p C_p \quad (1)$$

$$\frac{dC_p}{dt} = -\gamma C_p + \alpha N_p C_p - l_c C_p \quad (2)$$

Where,

N_p : Non-criminal minded population.

C_p : Criminal minded population.

μ : Natural growth rate.

α : Interaction rate between non-criminal minded and criminal minded population.

γ : Natural death rate.

l_c : Law enforcement rate.

K : Carrying capacity.

The third step is model modification, which is carried out by incorporating law enforcement and rehabilitation into the model. Next, in the fourth step, the optimal control problem is applied. At this stage, Pontryagin's Minimum Principle is used to find an analytical solution. Then, in the following step, the Forward-Backward Sweep method is employed in simulations to solve the problem numerically. Finally, the results of the numerical simulations are interpreted, and conclusions are drawn.

RESULT AND DISCUSSION

This section discusses two types of optimal control problems. In the first optimal control problem, law enforcement (u_1) is applied as the control variable to minimize the criminal minded population. In the second problem, both law enforcement (u_1) and rehabilitation (u_2) are used as control variables to minimize the criminal minded population and the cost of rehabilitation.

Optimal Control Problem I

The mathematical model to be optimized is represented by the system of equations (3)–(4), with the objective function given by equation (5).

$$\frac{dN_p}{dt} = \mu N_p\left(1 - \frac{N_p}{K}\right) - \alpha N_p C_p \quad (3)$$

$$\frac{dC_p}{dt} = -\gamma C_p + \alpha N_p C_p - u_1 C_p \quad (4)$$

$$\min J = \int_0^T C_p \, dt \quad (5)$$

By applying Pontryagin's Minimum Principle, equations (6)–(12) are obtained as follows.

Hamiltonian:

$$H = C_p(t) + \lambda_1\left(\mu N_p\left(1 - \frac{N_p}{K}\right) - \alpha N_p C_p\right) + \lambda_2\left(-\gamma C_p + \alpha N_p C_p - u_1 C_p\right) \quad (6)$$

State and costate equations:

It is assumed that the optimal values of $u_1(t)$, $N_p(t), C_p(t)$ and $\lambda(t)$ are denoted by $u^*(t), N_p^*(t), C_p^*(t)$ and $\lambda^*(t)$, respectively.

$$\left(\frac{dN_p}{dt}\right)^* = \left(\frac{\partial H}{\partial \lambda_1}\right)_* = \mu N_p^*\left(1 - \frac{N_p^*}{K}\right) - \alpha N_p^* C_p^* \quad (7)$$

$$\left(\frac{dC_p}{dt}\right)^* = \left(\frac{\partial H}{\partial \lambda_2}\right)_* = -\gamma C_p^* + \alpha N_p^* C_p^* - u_1^* C_p^* \quad (8)$$

$$\left(\frac{d\lambda_1}{dt}\right)^* = -\left(\frac{\partial H}{\partial N_p}\right)_* = -\left(\lambda_1^*\left(\mu - \frac{2\mu N_p^*}{K} - \alpha C_p^*\right) + \lambda_2^* \alpha C_p\right) \quad (9)$$

$$\left(\frac{d\lambda_2}{dt}\right)^* = -\left(\frac{\partial H}{\partial C_p}\right)_* = -\left(1 - \alpha\lambda_1^* N_p^* + \lambda_2^*\left(-\gamma + \alpha N_p^* - u_1^*\right)\right) \quad (10)$$

Optimal condition:

The optimal control obtained in this problem is a bang-bang optimal control, which is expressed using the signum function

$$u_1^*(t) = \begin{cases} u_{1_{max}} & if\ Q^*(t) < 0 \\ u_1(t)\epsilon[u_{1_{min}}, u_{1_{max}}] & if\ Q^*(t) = 0 \\ u_{1_{min}} & if\ Q^*(t) > 0 \end{cases} \quad (11)$$

where $Q^*(t) = C_p^* \lambda_2^*$ [20].

Since the type of system in this study is a free-final time and free-final state system, the boundary conditions are

$$\lambda_1(T) = 0 \text{ and } \lambda_2(T) = 0. \quad (12)$$

Optimal Control Problem II

The mathematical model being optimized is given by the system of equations (13)-(14), and the objective function is defined by equation (15), where $A > 0$ represents the preference weight for reducing the intervention cost.

$$\frac{dN_p}{dt} = \mu N_p\left(1 - \frac{N_p}{K}\right) - \alpha N_p C_p \quad (13)$$

$$\frac{dC_p}{dt} = -\gamma C_p + \alpha N_p C_p - u_1 C_p - u_2 C_p \quad (14)$$

$$\min J = \int_0^T C_p + A u_2^2\, dt \quad (15)$$

By applying Pontryagin's Minimum Principle, equations (16)-(23) are obtained as follows.

Hamiltonian:

$$H = C_p + Au_2^2 + \lambda_1\left(\mu N_p\left(1 - \frac{N_p}{K}\right) - \alpha N_p C_p\right) + \lambda_2\left(-\gamma C_p + \alpha N_p C_p - u_1 C_p - u_2 C_p\right) \quad (16)$$

State and costate equations:

It is assumed that the optimal values of $u_1(t), u_2(t), N_p(t), C_p(t)$ and $\lambda(t)$ are denoted by $u^*(t), N_p^*(t), C_p^*(t)$, and $\lambda^*(t)$, respectively.

$$\left(\frac{dN_p}{dt}\right)^* = \left(\frac{\partial H}{\partial \lambda_1}\right)_* = \mu N_p^*\left(1 - \frac{N_p^*}{K}\right) - \alpha N_p^* C_p^* \quad (17)$$

$$\left(\frac{dC_p}{dt}\right)^* = \left(\frac{\partial H}{\partial \lambda_2}\right)_* = -\gamma C_p^* + \alpha N_p^* C_p^* - u_1^* C_p^* - u_2^* C_p^* \quad (18)$$

$$\left(\frac{d\lambda_1}{dt}\right)^* = -\left(\frac{\partial H}{\partial N_p}\right)_* = -\left(\lambda_1^*\left(\mu - \frac{2\mu N_p^*}{K} - \alpha C_p^*\right) + \lambda_2^* \alpha C_p\right) \quad (19)$$

$$\left(\frac{d\lambda_2}{dt}\right)^* = -\left(\frac{\partial H}{\partial C_p}\right)_* = -\left(1 - \alpha\lambda_1^* N_p^* + \lambda_2^*\left(-\gamma + \alpha N_p^* - u_1^* - u_2^*\right)\right) \quad (20)$$

Optimal conditions:

The optimal control $u_1^*(t)$ is of bang-bang type, meaning it only takes on extreme values—either the minimum or the maximum.

$$u_1^*(t) = \begin{cases} u_{1_{max}} & if\ Q^*(t) < 0 \\ u_{1_{min}} & if\ Q^*(t) > 0 \end{cases} \quad (21)$$

where $Q^*(t) = C_p^* \lambda_2^*$.

Whereas

$$u_2^*(t) = \frac{\lambda_2^* C_p^*}{2A} \quad (22)$$

where $0 \leq u_2^* \leq u_{2_{max}}$.

Since the type of system in this study is a free-final time and free-final state system, the boundary conditions are given by

$$\lambda_1(T) = 0 \text{ and } \lambda_2(T) = 0. \quad (23)$$

The system of equations obtained by applying Pontryagin's Minimum Principle is then implemented in a numerical simulation. The system of state and costate differential equations is solved using the Forward-Backward Sweep method based on the fourth-order Runge-Kutta scheme. The state equations are solved using the forward method, while the costate equations are solved using the backward method.

Numerical Simulations

The parameter values used in the numerical simulation are $\alpha = 0.5$, $\mu = 1.3$, $\gamma = 1.9$, dan $K = 6$, with the initial population conditions set as $N_p(0) = 1$, and $C_p(0) = 1$ [14]. The simulation results shown in Figure 3 compare the mathematical model without control and with optimal control in Problem I. In Figure 3, it is evident that optimal control can significantly reduce the criminal minded population. Without intervention, the number of criminal minded individuals decreases slightly from the initial value $C_p(0) = 1$ to $C_p(T) = 0.953$, which is due to the natural death rate within the criminal minded population. However, with optimal control—specifically the implementation of law enforcement—the final value decreases to $C_p(T) = 0.086$. The profile of optimal control, i.e. law enforcement, is shown in Figure 4. From Figure 4, it can be observed that law enforcement is applied at full intensity (100%) for a specific period of time until it successfully suppresses the criminal minded population C_p and brings the system to equilibrium.

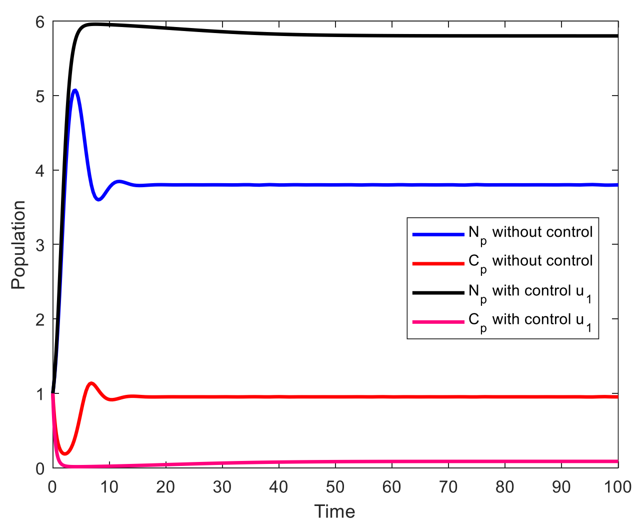

Figure 3. Number of criminal and non-criminal minded population over time in Problem I

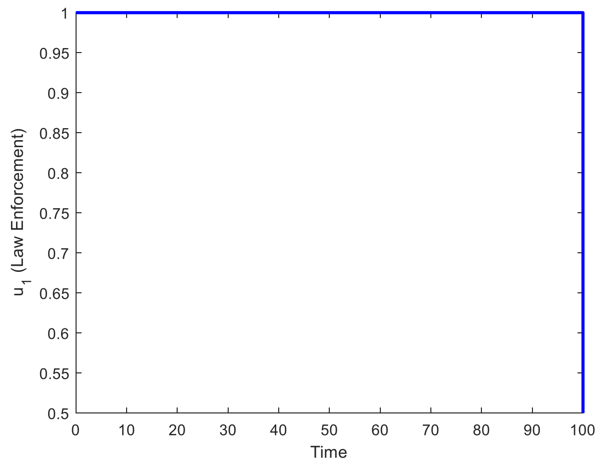

Figure 4. Law enforcement profile as optimal control on Problem I

Figure 5 presents the results of a comparison between the model without optimal control and the model with control in Optimal Control Problem II. In Figure 5, it is shown that the applied optimal control successfully reduces the criminal population significantly. By implementing law enforcement and rehabilitation on the criminal population, the number is reduced from $C_p(0) = 1$ to $C_p(T) = 0.012$. The profiles of the control variables are shown in Figures 6 and 7. In Figure 6, it can be seen that law enforcement is applied at full capacity (100%) up to a certain time period, until it effectively suppresses the criminal minded population C_p and brings it to equilibrium. In contrast, Figure 7 shows that rehabilitation is not applied at full intensity. This is due to the inclusion of rehabilitation cost minimization in the objective function. Meanwhile, the cost of law enforcement is not minimized in the model.

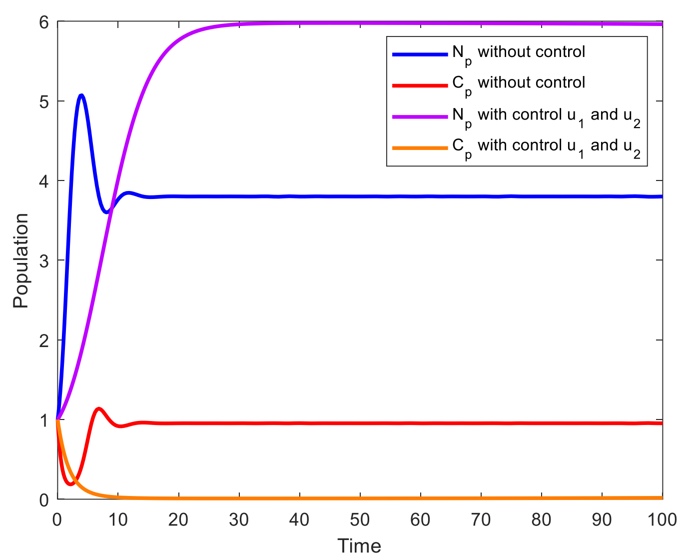

Figure 5. Number of criminal and non-criminal population over time in Problem II

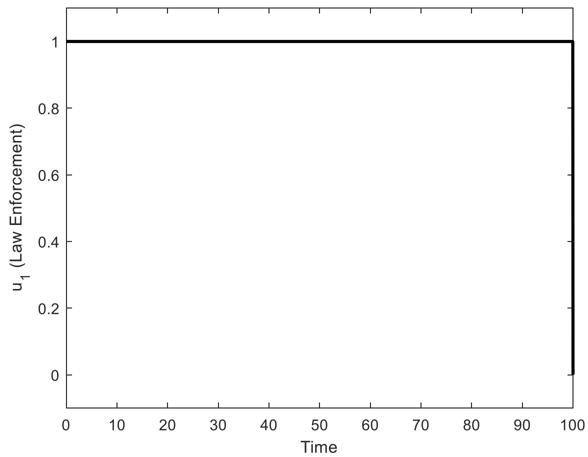

Figure 6. Law enforcement profile as optimal control on Problem II

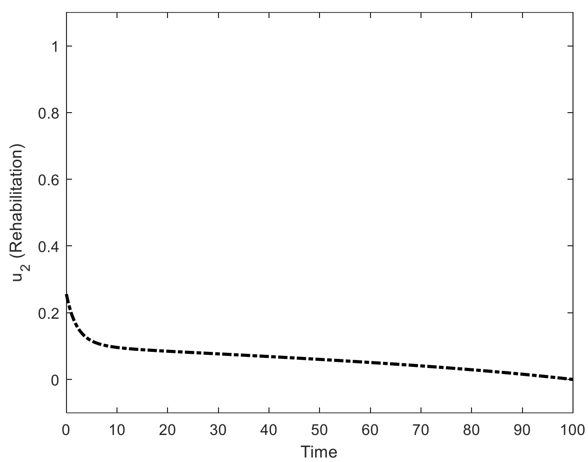

Figure 7. Rehabilitation profile as optimal control in Problem II

Table 1. Comparison of the Number of Non-Criminal Population and Criminal Population at Equilibrium Conditions

Number Population	Without Optimal Control	Optimal Control Problem I	Optimal Control Problem II
Non-Criminal minded	3.799	5.801	5.972
Criminal minded	0.953	0.086	0.012

The comparison of population size over time with and without optimal control is presented in Table 1. The results show that the number of criminal individuals under Optimal Control Problem II is smaller than in Problem I. This indicates that the application of both law enforcement and rehabilitation is more effective in minimizing the criminal population than law enforcement alone. This finding aligns with the statement in the introduction, which emphasizes that restorative measures—in this case, rehabilitation programs—are needed to address criminal behavior, alongside repressive measures, namely law enforcement.

CONCLUSION

Based on the results obtained, it can be concluded that the formulated optimal control problem can minimize the criminal population in the mathematical model of social behavior. This study only incorporated rehabilitation as a factor in addressing criminal acts. Future research may modify the model by considering other factors for intervention or by including additional state variables.

REFERENCE

[1] D. Guntara dan Budiman, "Tinjauan Kriminologi Terhadap Pelaku Tindak Pidana Terorisme di Indonesia dalam Perspektif Teori Differential Association," *Jurnal Justisi Hukum,* vol. 3, no. 1, pp. 106-119, 2018.

[2] F. Ramadhan dan I. Ridwan, "Analisa Kasus Kekerasan Di STPDN Ditinjau Dari Teori Differential Association Edwin H Sutherland Dalam Hukum Pidana Di Indonesia," *ATTAQWA: Jurnal Pendidikan Islam dan Anak Usia Dini,* vol. 1, no. 1, pp. 12-18, 2022.

[3] N. F. A. Idrus dan Y. Widowati, "Cyberbullying di Media Sosial dalam Pres Kriminologis dan Viktmologis," *Diversi Jurnal Hukum,* vol. 8, no. 2, pp. 217-241, 2022.

[4] C. J. Hisyam, A. Miftaqiyah, A. A. H. Putra, C. A. Kinanti, D. N. Ardani, N. P. Lubis dan S. Adila, "Analisis Pelaku Kenakalan Remaja "Klitih" dalam Perspektif Teori Asosiasi Diferensial Sutherland," *Harmoni : Jurnal Ilmu Komunikasi dan Sosial,* vol. 1, no. 4, pp. 81-89, 2023.

[5] C. Timur dan L. Nurhadiyanto, "Tinjauan Penggunaan Rokok Elektrik di Kalangan Remaja dalam Perspektif Teori Differential Association Theory," *Action Research Literate,* vol. 8, no. 8, pp. 2229-2233, 2024.

[6] B. Cahyadi, E. G. Gara, P. Pratama, G. Fitriadi, Arwansa dan D. S. Arian, "Hacker Anak Dalam Perspektif Teori Differential Association: Studi Kasus Peretasan Situs Pengadilan Negeri Kabupaten Konawe," *Ikraith-Humaniora,* vol. 8, no. 1, pp. 329-340, 2024.

[7] C. J. Hisyam, A. Aditya, D. Az-Zahra, D. Galih, F. Sholihah, M. Nafilata dan E. S. Sahla, "Keterasingan Sosial Sebagai Faktor Pemicu Penyalahgunaan Narkoba Di Kalangan Remaja," *Jurnal Ilmiah Research and Development Student,* vol. 3, no. 1, pp. 122-135, 2025.

[8] J. Sooknanan, B. Bhatt dan D. M. G. Comissiong, "A modified predator–prey model for the interaction of police and gangs," *Royal Society Open Science,* vol. 3, no. 160083, pp. 1-15, 2017.

[9] J. O. Akanni, F. O. Akinpelu, S. Olaniyi, A. Oladipo dan A. Ogunsola, "Modelling financial crime population dynamics: optimal control and cost-effectiveness analysis," *International Journal of Dynamics and Control,* vol. 8, p. 531–544, 2019.

[10] W. Nur dan Darmawati, "Mathematical Model of Armed Criminal Group with Pre-emitive and Repressive Intervention," *JOMTA Journal of Mathematics: Theory and Applications,* vol. 2, no. 2, pp. 27-32, 2020.

[11] A. Wahid, S. Toaha dan Kasbawati, "Kontrol Optimal Model Matematika Dinamika Korupsi dengan Pemberian Edukasi dan Kampanye, Perbaikan Sistem, dan Represif," *Proximal,* vol. 6, no. 1, pp. 53-68, 2022.

[12] H. A. Ashi, "Stability analysis of a simple mathematical model for school bullying," *AIMS Mathematics,* vol. 7, no. 4, p. 4936–4945, 2022.

[13] C. Alfiniyah, A. Puspitasari dan Fatmawati, "Mathematical Modelling of Drug Abuse Reduction Strategies taking into account the Treatment Type and Risks Level," *Jambura Journal of Biomathematics,* vol. 4, no. 1, pp. 23-30, 2023.

[14] S. Abbas, J. P. Tripathi dan A. A. Neha, "Dynamical analysis of a model of social behavior: Criminal vs non-criminal population," *Chaos, Solitons, and Fractals,* vol. 98, pp. 121-129, 2017.

[15] A. K. Srivastav, M. Gosh dan P. Chandra, "Modeling dynamics of the spread of crime in a society," *Stochastic Analysis and Applications,* vol. 37, no. 6, pp. 991-1011, 2019.

[16] J. P. Tripathi, S. Bugalia, K. Burdak dan S. Abbas, "Dynamical analysis and effects of law enforcement in a social interaction model," *Physica A,* vol. 567, p. 125725, 2021.

[17] M. Kumar dan S. Abbas, "Modelling and prevention of crime using age-structure and law enforcement," *Journal of Mathematical Analysis and Applications,* vol. 519, no. 2, p. 126849, 2023.

[18] M. Kumar, S. Dhama, F. A. Alqarni dan S. Abbas, "Mathematical analysis and optimal control of age-structured social interaction model with law enforcement," *Mathematical Methods in the Applied Sciences,* vol. 48, no. 3, pp. 3712-3725, 2024.

[19] N. Izzati, J. W. Leksono dan N. Yannuansa, "Dynamical Analysis of Mathematical Model of Social Behavior with Law Enforcement and Religious Approaches," *J Statistika: Jurnal Ilmiah Teori dan Aplikasi Statistika,* vol. 17, no. 1, pp. 672-681, 2024.

[20] W. Ningsih, H. Purwaningsih dan R. El Maghfiroh, "Kendali Optimal Model Pertumbuhan Mikroalga dalam Chemostat," *Limits: Journal of Mathematics and Its Applications,* vol. 19, no. 2, pp. 229-240, 2022.